10-3 Powers and Roots of Complex Numbers

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Multiplying Complex Numbers in Modulus-Argument Form

$$z = r(\cos\theta + i\sin\theta)$$

$$z^2 = [r(\cos\theta + i\sin\theta)]^2$$

$$z^2 = [r(\cos\theta + i\sin\theta)]\cdot[r(\cos\theta + i\sin\theta)]$$

Using the formula from the previous section:

$$\boxed{z_1 \cdot z_2 = r_1 \cdot r_2 \cdot [(\cos(\theta_1 + \theta_2) + i(\sin\theta_1 + \theta_2)]}$$

$$z^2 = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)]$$

$$z^2 = r^2[\cos(2\theta) + i\sin(2\theta)]$$

And now multiply z^2 by z to find z^3

$$z^3 = r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)]$$

$$z^3 = r^3[\cos(3\theta) + i\sin(3\theta)]$$

And so on....

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DeMoivre's Theorem

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

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Ex1. Find

1.) $(1 + i)^5$ $z = \sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$

$\sqrt{2}^{\,5}\ \text{cis}\left(\frac{5 \cdot \pi}{4}\right)$

2.) $(-\sqrt{2}+i\sqrt{2})^{-2}$ $r = \sqrt{2+2} = 2$ $\theta = \tan^{-1}(-1)$

$\theta = -\frac{\pi}{4}$

$\theta = \frac{3\pi}{4}$

$\left[2\,\text{cis}\left(\frac{3\pi}{4}\right)\right]^{-2}$

$(2)^{-2}\,\text{cis}\left(\frac{-6\pi}{4}\right)$

$= \frac{1}{4}\,\text{cis}\left(-\frac{3\pi}{2}\right)$

Ex2. Simplify

$$\frac{\left(\cos(8\theta)+i\sin(8\theta)\right)\cdot\left(\cos(5\theta)+i\sin(5\theta)\right)}{\left(\cos(\theta)+i\sin(\theta)\right)}$$

$$\frac{cis(8\theta)\cdot cis(5\theta)}{cis\,\theta}$$

$$cis((8+5-1)\theta)=cis(12\theta)$$

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Remember

$x^3 - 8 = 0$

$x^3 = 8$

$x = 2$

so

$x^3 - 8 = (x-2)\cdot g(x)$

$x^3 - 8 = (x-2)\cdot\left(x^2 + 2x + 4\right)$

so

$$\begin{array}{r} x^2 + 2x + 4 \\ x-2\,\overline{)\,x^3 + 0x^2 + 0x - 8} \\ -\underline{\left(x^3 - 2x^2\right)} \\ 2x^2 + 0x \\ -\underline{\left(2x^2 - 4x\right)} \\ -4x - 8 \\ -\underline{\left(-4x - 8\right)} \\ 0 \end{array}$$

$x = \dfrac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 3}}{2\cdot 1}$

$x = \dfrac{-2 \pm \sqrt{-12}}{2}$

$x = \dfrac{-2 \pm 2i\sqrt{3}}{2}$

$x = -1 \pm i\sqrt{3}$

$\boxed{x = 2, -1 \pm i\sqrt{3}}$

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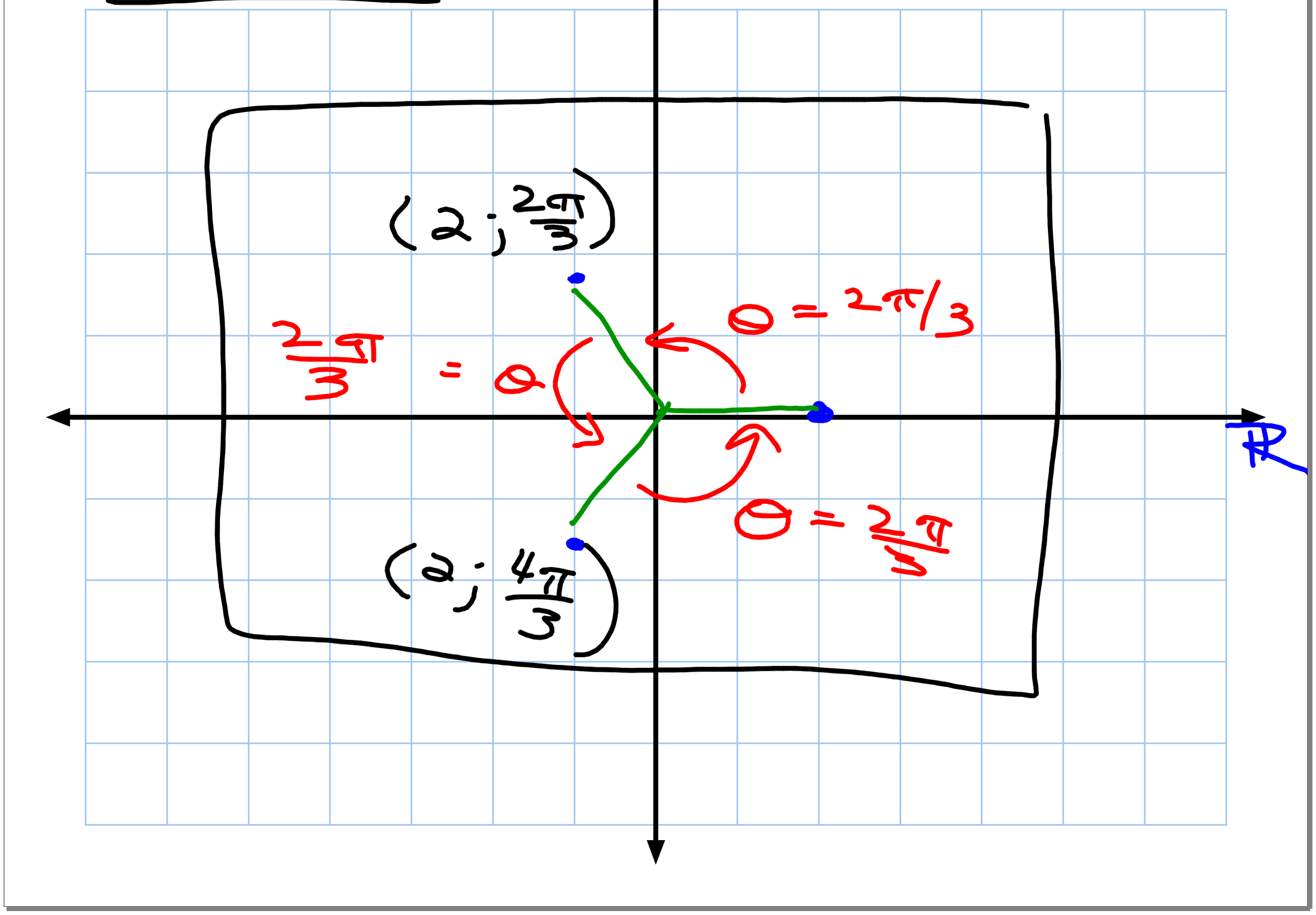
$(-1 \pm \sqrt{3}\,i)$



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Note: $2^3 - 8$ BUT

$$(-1+i\sqrt{3})^3$$
$$= (-1+i\sqrt{3})\cdot(-1+i\sqrt{3})\cdot(-1+i\sqrt{3})$$
$$= (1-2i\sqrt{3}+i^2\sqrt{9})\cdot(-1+i\sqrt{3})$$
$$= (1-2i\sqrt{3}-3)\cdot(-1+i\sqrt{3})$$
$$= (-2-2i\sqrt{3})\cdot(-1+i\sqrt{3})$$
$$= 2-2i\sqrt{3}+2i\sqrt{3}-2i^2\sqrt{9}$$
$$= 2+6=8$$

$$(-1-i\sqrt{3})^3$$
$$= (-1-i\sqrt{3})\cdot(-1-i\sqrt{3})\cdot(-1-i\sqrt{3})$$
$$= (1+2i\sqrt{3}+i^2\sqrt{9})\cdot(-1-i\sqrt{3})$$
$$= (1+2i\sqrt{3}-3)\cdot(-1-i\sqrt{3})$$
$$= (-2+2i\sqrt{3})\cdot(-1-i\sqrt{3})$$
$$= 2+2i\sqrt{3}-2i\sqrt{3}-2i^2\sqrt{9}$$
$$= 2+6=8$$

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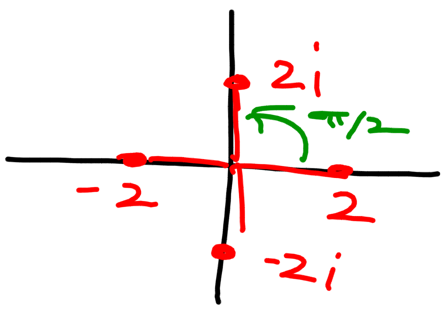
Ex4. Find all four roots of $\sqrt[4]{16}$



$x^4 = 16$

$\sqrt[4]{x^4} = \sqrt[4]{16}$

$x = \pm 2$

$(2i)^4$

$\underbrace{(2i)(2i)}_{-4}\underbrace{(2i)(2i)}_{-4}$

$-4 \cdot -4 = 16$

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The Distinct nth Roots of a Complex Number

Let $z = r(\cos\theta + i\sin\theta)$ and let $n \in \mathbb{Z}^+$, then *z* has *n* distinct roots

$$z^k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \cancel{\frac{2\pi k}{n}}\right) + i\sin\left(\frac{\theta}{n} + \cancel{\frac{2\pi k}{n}}\right)\right]$$

~~Where k = 0, 1, 2, 3, 4, ..., n-1~~.

Note – each of the nth roots of z have the same modulus $\sqrt[n]{r}$. Thus all these points lie on a circle in the complex plane and the roots are equally spaced on the circle.

The principle root corresponds to the k=0 iteration of the formula.

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$+\pi$

Ex5. Find the 3 cube roots of -2 – 2i and graph them in the complex plane.

$r = \sqrt{(-2)^2 + (-2)^2}$ $\qquad \tan^{-1}\left(\frac{-2}{-2}\right) = \tan^{-1}(1)$

$r = \sqrt{4+4}$ $\qquad = \pi/4$

$r = \sqrt{8}$ $\qquad 5\pi/4$

$2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{4}\right)$ $\qquad r = 2\sqrt{2}$

$\theta = 5\pi/4$

$n = 3$

$K=0 \quad \left(8^{1/2}\right)^{1/3} \text{cis}\left(\frac{5\pi/4}{3}\right)$

$8^{1/6}\ \text{cis}(5\pi/12)$

$\left(2^3\right)^{1/6} \text{cis}\left(\frac{5\pi}{12}\right)$

$\boxed{\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)}$

$K=1:$

$\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12} + \frac{2\pi}{3}\right)$

$\sqrt{2}\,\text{cis}\left(\frac{13\pi}{12}\right)$

$K=2:$ $\quad \sqrt{2}\,\text{cis}\left(\frac{5\pi}{12} + \frac{4\pi}{3}\right)$

$\boxed{\sqrt{2}\,\text{cis}\,\frac{21\pi}{12}}$

Ex6. Find the 5th roots of $3+i\sqrt{3}$.

$3+i\sqrt{3} \quad r=\sqrt{3^2+3}=\sqrt{12}=2\sqrt{3} \quad \theta=\pi/6$

$2\sqrt{3}\,\mathrm{cis}\,\pi/6$

$\boxed{\sqrt[5]{2\sqrt{3}}\left(\cos\frac{\pi}{30}+i\sin\frac{\pi}{30}\right)}$

$\sqrt[5]{2\sqrt{3}}\left(\cos\left(\frac{\pi}{30}+\frac{2\pi}{5}\right)+i\sin\left(\frac{\pi}{30}+\frac{2\pi}{5}\right)\right)$

$\boxed{\sqrt[5]{2\sqrt{3}}\ \mathrm{cis}\left(\frac{13\pi}{30}\right)}$

$\sqrt[5]{2\sqrt{3}}\ \mathrm{cis}\,\frac{25\pi}{30}=\boxed{\sqrt[5]{2\sqrt{3}}\ \mathrm{cis}\,\frac{5\pi}{6}}$

$\boxed{\sqrt[5]{2\sqrt{3}}\ \mathrm{cis}\,\frac{37\pi}{30}}$

$\boxed{\sqrt[5]{2\sqrt{3}}\ \mathrm{cis}\,\frac{49\pi}{30}}$

Taylor Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots. + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots. + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots. + \frac{x^n}{n!} + \ldots$$

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$$\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \ldots$$

Now this looks a lot like the expansion of f(x) = e^x except for the sign change discrepancies. The signs alternating in such a way where pairs of terms alternate seems like it has something to do with the powers of i.

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

$i^7 = -i$

$i^8 = 1$

This suggests expanding $f(x) = e^{ix}$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \frac{(ix)^9}{9!} + \dots + \frac{(ix)^n}{n!} + \dots$$

$$e^{ix} = 1 + xi - \frac{x^2}{2!} - \frac{x^3}{3!}i + \frac{x^4}{4!} + \frac{x^5}{5!}i - \frac{x^6}{6!} - \frac{x^7}{7!}i + \frac{x^8}{8!} + \frac{x^9}{9!}i + \dots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6} + \frac{x^8}{8!} + \dots\right) + \left(xi - \frac{x^3}{3!}i + \frac{x^5}{5!}i - \frac{x^7}{7!}i + \frac{x^9}{9!}i + \dots\right)$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{8!} + \dots\right)$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

$\therefore$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$t=\cos(x)+i\sin(x)$$

$$\frac{dt}{dx}=\frac{d}{dx}(\cos(x)+i\sin(x))=\frac{d}{dx}\cos(x)+\frac{d}{dx}i\sin(x)$$

$$\frac{dt}{dx}=-\sin(x)+i\cos(x)$$

$$-1=i^2$$

$$\frac{dt}{dx}=i^2\sin(x)+i\cos(x)$$

$$\frac{dt}{dx}=i(i\sin(x)+\cos(x))$$

$$\frac{dt}{dx}=it$$

$$\frac{1}{t}\frac{dt}{dx}=i$$

$$\int\frac{1}{t}\,dt=\int i\,dx$$

$$ln|t|=ix+C$$

$$ln|\cos(x)+i\sin(x)|=ix+C$$

$$ln|\cos(0)+i\sin(0)|=i0+C$$

$$ln|1+i0|=i0+C$$

$$ln|1|=C$$

$$0=C$$

$$ln|\cos(x)+i\sin(x)|=ix$$

$$e^{xi}=\cos(x)+i\sin(x)$$

Euler's Formula

Any complex number $z = r\left(\cos\theta + i\sin\theta\right)$ can be expressed as $re^{i\theta}$.

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$re^{i\theta} = r(\cos\theta + i\sin\theta)$

Ex7. Evaluate

1.) $e^{i\pi} = \cos\pi + i\sin\pi$
$= -1 + i \cdot 0 = \boxed{-1}$

2.) $5e^{\pi/3\, i}$
$5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$
$5\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{5}{2} + \frac{5i\sqrt{3}}{2}$

Ex8. Express in Exponential Form

1.) $3\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)$

$3e^{i\pi/3}$

2.) $8+8i$

$r=\sqrt{8^2\cdot 2}=8\sqrt{2}$

$\theta=\tan^{-1}(1)=\frac{\pi}{4}$ (1st Quad)

$8\sqrt{2}\,e^{i\frac{\pi}{4}}$

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Ex9. Evaluate $(1-i\sqrt{3})^8$

$r=\sqrt{1+3}=2$

$\theta=\tan^{-1}(-\sqrt{3})=\frac{-\pi}{3}$

$\left(2e^{-\frac{\pi}{3}i}\right)^8$

$2^8 \cdot e^{-\frac{8\pi i}{3}}$

$2^8 \operatorname{cis}\left(\frac{-8\pi}{3}\right)$

$2^8\left[\cos\left(\frac{-8\pi}{3}\right)+i\sin\left(\frac{-8\pi}{3}\right)\right]$

Ex10. Simplify

$$\frac{(\cos(8\theta)+i\sin(8\theta))(\cos(5\theta)+i\sin(5\theta))}{(\cos(\theta)+i\sin(\theta))}$$

$$\frac{e^{8\theta i}\cdot e^{5\theta i}}{e^{\theta i}} = \frac{e^{13i\theta}}{e^{i\theta}} = e^{12i\theta}$$

$$\text{cis}(12\theta)$$

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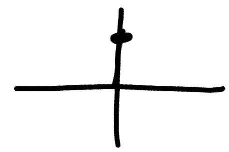
Ex11. Use Euler's formula to find the 4th roots of i. $0 + 1i$



$r = \sqrt{0^2 + 1^2} = 1 \qquad \theta = \frac{\pi}{2}$

$1 \operatorname{cis} \pi/2 \Longrightarrow e^{\pi/2 i}$

$\frac{\pi}{2} = \frac{4\pi}{8}$

$\left(e^{\pi/2 i}\right)^{1/4} = e^{\pi/8 i} \Longrightarrow$

$$\boxed{\begin{array}{l} \operatorname{cis} \frac{\pi}{8} \\ \operatorname{cis} 5\pi/8 \\ \operatorname{cis} 9\pi/8 \\ \operatorname{cis} 13\pi/8 \end{array}}$$

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$$(i)^i = \left(e^{\frac{\pi}{2}i}\right)^i$$

$$i^i = e^{\frac{\pi}{2}i^2} \quad i^2 = -1$$

$$i^i = e^{-\frac{\pi}{2}}$$

$$\operatorname{cis}\left(-\frac{\pi}{2}\right)$$

$$\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

$$0 + -1i$$

$$i^i = -1i$$

HW pg 458 #1, 3, 6, 7, 9, 10, 16, 20, 21, 24, 26, 28, 30-33, 37-42, 45, 47, 49

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